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1. The Problem and its History

Henry the 8th looks for a new spouse

Image communication is an old problem...

Anne de Clève, Holbein, 1539
How many bits for Mona Lisa?

<=> \{0,1\}
2. Mathematical Representation of Signals

Joseph Fourier (1768-1830)
Studies the heat equation (in Egypt...)

Brice Lecture - 5
1807: Fourier upsets the French Academy....

Fourier Series:
- Harmonic series
- Frequency changes, $f_0, 2f_0, 3f_0, \ldots$
“What is the magic trick?”

or, with Euclid:

orthogonality

coordinate system

and: successive approximation
But

1898: Gibbs’ paper  
1899: Gibbs’ correction

and it will take almost another 60 years to settle the convergence question (Carleson 1964).
1910: Alfred Haar discovers the Haar wavelet dual to the Fourier construction

Haar series:
- Scale change, scales $S_0, 2S_0, 4S_0, 8S_0$
- Time shift
The Haar system

Again a set of orthonormal vectors!

Size: length proportional to $2^m$

“frequency” : $f_0$, $2f_0$, $4f_0$, $8f_0$, ... octaves!
1945: Gabor localizes the Fourier transform $\Rightarrow$ STFT

1980: Morlet proposes the continuous wavelet transform

short-time Fourier transform wavelet transform
Analogy with the musical score
Bach knew about wavelets!
1983: Lena discovers pyramids (actually, Burt and Adelson)
1984: Lena gets critical (subband coding)
1986: Lena gets formal...
(multiresolution theory by Mallat, Meyer...)

= +
1988: Ingrid discovers Daubechies’ wavelets!

- New families of orthonormal bases, (generalizing Haar)
- Biorthogonal families, frames
- many new applications
3. Information Theory, Signal Processing and Wavelets

Claude Shannon: The founding genius

1. Source coding
2. Channel coding
3. Separation of source and channel coding
Source Coding

exchanging description complexity for quality

Again, successive approximation is key
**Signal Processing**

**Subband coding**

![Diagram of subband coding with analysis and synthesis processes, including filters H0, H1, G0, G1, and a beam of white light.]
Iterated filter banks

stage 1

H_0 \downarrow 2

H_1 \downarrow 2

stage 2

H_0 \downarrow 2

H_1 \downarrow 2

stage J

H_0 \downarrow 2

Frequency division

|H_i|

\ldots \frac{\pi}{16} \frac{\pi}{8} \frac{\pi}{4} \frac{\pi}{2} \pi

Frequency division
Separable application in 2D

An image and its wavelet decomposition

Important:
• auditory system works in octaves
• visual system works in frequency bands
The iterated filter bank leads to wavelets

The Daubechies iterative wavelet construction

Scaling function and Wavelet
Finite length, continuous $\varphi(t)$ and $\psi(t)$, based on $L=4$ iterated filter
Iterated filter banks lead to two-scale equations

\[ \phi(t) = \sum_{n} c_n \phi(2t - n) \]

Hat function

Daubechies' scaling function

Relation to self-similarity useful for analysis and characterization of fractal processes
4. Wavelets and Approximation Theory

Consider piecewise smooth signals

- Wavelets act as singularity detectors
- Scaling functions catch smooth parts
- “Noise” is circularly symmetric
How does this work? Proper choice of filters!

Iterated filter bank \( (H_j(z) = G_j(z^{-1})) \)

- polynomials are “eaten” in the highpass
- polynomials are reproduced by the lowpass channel
- discontinuities are detected by the wavelets
Example: $S_4$ reproduces linear fcts
How about singularities?

If we have a singularity of order \( n \) at the origin (-1 Dirac, 0: Heaviside,...), the CWT transform behaves as

\[
X(a, 0) = c_n \cdot a^{n/2}
\]

In the orthogonal wavelet series: same behavior, but only \( L=2N-1 \) coefficients influenced at each scale!

- e.g. Dirac/Heaviside: behavior as \( 2^{-m/2} \) and \( 2^{m/2} \)
Example:

- phase changes randomize signs, but not decay
- a singularity influences only $L$ wavelets at each scale ($L=2N-1=3$)
Approximation: linear versus non-linear

Given an orthonormal basis \( \{g_n\} \) for a space \( S \) and a signal

\[
f = \sum_n \langle f, g_n \rangle \cdot g_n,
\]

- the best linear approximation is given by the projection onto a fixed subspace of size \( M \) \((\text{independent of } f!\))

\[
\hat{f}_M = \sum_{n=1}^{M} \langle f, g_n \rangle \cdot g_n
\]

- the best nonlinear approximation is given by the projection onto an adapted subspace of size \( M \) \((\text{dependent on } f!)\)

\[
f_M = \sum_{n \in I_M} \langle f, g_n \rangle \cdot g_n \quad \rightarrow \quad I_M: \text{set of largest } M \text{ coeffs}
\]

or: take the first \( M \) coeffs (linear) or take the largest \( M \) coeffs (non-linear)
Nonlinear approximation

Nonlinear approximation power depends on basis

Example:

Two different bases for \([0,1]\):

- Fourier series \(\{e^{j2\pi kt}\}_{k \in \mathbb{Z}}\)
- Wavelet series: Haar wavelets

Linear approximation in Fourier or wavelet bases

\[ \hat{\varepsilon}_M \sim 1/M \]

Nonlinear approximation in a Fourier basis

\[ \tilde{\varepsilon}_M \sim 1/M \]

Nonlinear approximation in a wavelet basis

\[ \tilde{\varepsilon}_M \sim 1/2^M \]
Fourier Basis: N=1024, M= 64, linear versus nonlinear

- nonlinear approximation is not necessarily much better!
Wavelet basis: N=1024, M=64, J=6, linear versus nonlinear

- nonlinear approximation is vastly superior!
5. Approximation and Applications in Denoising and Compression

Wavelets approximate piecewise smooth signals with few non-zero coefficients

This is good for
- Compression
- Denoising
- Classification
- Inverse problems

Thus: sparsity is good!
Denoising

Idea:
- Dominant features are caught by large wavelet coefficients
- Noise is spread uniformly over all coefficients
- Thresholding small coefficients to 0 keeps the signal but removes the noise

Schematically:

\[
x[n] + w[n] \xrightarrow{WT} [\hat{x}[n]] \xrightarrow{\text{Thresholding}} \hat{w}[n] \xrightarrow{\text{IWT}} \hat{x}[n] + \hat{w}[n]
\]

Now:
\[\hat{x}[n] \sim x[n]\]
\[|\hat{w}[n]| \ll |w[n]|\]

Note:
- very simple
- works well for piecewise smooth signals
- for jointly gaussian, standard linear methods (Wiener filter) are fine
Example: 1D Signal
Example: 2D signal
Compression

Idea

- sparse representation should be good for compression
- transform, keep large coefficients through quantization
- reconstruction gives good quality

Note

- simple
- at the heart of JPEG 2000
- for jointly Gaussian, standard linear approach (KLT) is optimal
Example: 1D

WT

Quantization

IWT
Example: 2D

WT

Quantization

IWT
**Notes**

- improvement by a few dB’s
- lot more functionalities (e.g. progressive download on internet)
- low rate behavior
- is this the limit?
From the comparison, JPEG fails above 40:1 compression while JPEG2000 survives.

Images courtesy of www.dspworx.com
So, are wavelets closing the “How many bits for Mona Lisa” question?

(un) fortunately: No!

Reason:

Shannon tells us

\[ D(R) \sim \alpha_1 2^{-\beta_1 R} \]

but wavelets give

\[ D_W(R) \sim \alpha_2 \sqrt{R} 2^{-\beta_2 \sqrt{R}} \]

for certain classes of simple signals
Reason: independent coding of dependent information

All these wavelets coefficients correspond to a single degree of freedom!
Solution: model dependencies between wavelets coefficients

Various proposals

- Markov models (Baraniuk)
- Zero trees
- Footprints
Example: An optimal algorithm

This uses dynamic programming [Prandoni:00]
Wavelet Footprints [Dragotti:01]

Can we “fix” the wavelet scenario?

That is, achieve the same rate-distortion performance as an oracle or a dynamic programming method but with the simplicity of wavelet methods?

The structure of wavelet representation of singularities is simple:

- location: random
- structure across scales: deterministic!

Data structure to capture discontinuities in wavelet domain

- in orthogonal expansion
- in frame

This leads to a simple and intuitive data structure

Wavelet Footprint
The wavelet footprint

- this is the signature of the discontinuity
- behaviour well understood (classic wavelet analysis)
Denoising

Original signal

Noisy Signal (SNR=15.62dB)

Hard-Thresholding (SNR=21.3dB)

Cycle-Spinning (SNR=25.4dB)

Denoising with Footprints (SNR=27.2dB)
Compression

Original signal

Compressed Fp. approx.

Residue

Residue compressed

Fp. compr. SNR=17.2dB 0.33b/p

SPITH SNR=14.3dB 0.33b/p
6. Going to Two Dimensions: Nonseparable Bases

Objects in two dimensions we are interested in

• textures: \( D(R) = C_0 \cdot 2^{-2^R} \) per pixel
• smooth surfaces: \( D(R) = C_1 \cdot 2^{-2^R} \) per object!
Current approaches to two dimensions....

Mostly separable, direct products

Wavelets: good for point singularities
but what is needed are sparse coding of edge singularities!
Two dimensional wavelet bases

Ex: Tensor products of Haar functions

That is

or in 3D

That is very little directionality!
What is needed are directional bases

- Local Radon transform
- Ridgelets
- Curvelets
- Contourlets
- etc

That is:

a zoo of true two dimensional animals
Example: a directional block transform [Do:01]
Consider object $c^2$ boundary between two cst

- # of wavelet coeffs: $2^j$
- # of curvelet coeffs: $2^{j/2}$

Rate fo approximation, M-term NLA

- Fourier: $O(1/\sqrt{M})$
- Wavelets: $O(1/M)$
- Curvelets: $O(1/M^2)$
Operational Solution

Directional Analysis (as in Radon transform)
+ Multiresolution as in wavelets

Directional Filter Banks

• division of 2-D spectrum into fine slices using iterated tree structured filter banks
Pyramidal Directional Filter Banks (PDFB)

Motivation: + add multiscale into the directional filter bank
+ improve its non-linear approximation power

Properties: + Flexible multiscale and directional representation for images (can have different number of direction at each scale!)
Example: A pyramidal directional filter bank

Compression, denoising, inverse problems: mostly open!
7. Conclusions

Multiresolution is good for you!
  • Perception and mathematics (mostly) agree...

Non-linear can buy a lot...
  • in approximation, the difference can be huge!

Compression is hard but generic
  • understanding complexity is fundamental

Multiple dimension is (infinitely) harder than one...

The search for the ultimate basis is a fascinating and timeless topic
References

For a tutorial:

M. Vetterli, Wavelets, Approximation and Compression, Signal Processing, May 2001

For more details


M. Do, M. Vetterli, Pyramidal Directional Filter Banks, to be submitted, 2002
Appendix:

1930: Heisenberg discovers that you cannot have your cake and eat it too!

Uncertainty principle
  • lower bound on TF product

Time-frequency tiling for a
\[ x[n] = \cos 2\pi f_0 + A \delta[n - t_0] \]

so....

what is a good basis?