Sampling Signals of Finite Rate of Innovation*

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1 Motivation

Signal Processors love bandlimited Signals...

Then:

\[ \{f(nT)\}, n \in \mathbb{Z}, T = \frac{\pi}{\omega_m} \]

is a sufficient representation, since

\[ f(t) = \sum_{n \in \mathbb{Z}} f(nT) \sin c\left(\frac{t}{T} - n\right) \quad (1) \]

where

\[ \sin c(t) = \frac{\sin(\pi t)}{\pi t} \quad \mathcal{F} \quad I[-\pi, \pi] \]
But what if

just one discontinuity and no more sampling theorem...

Often, one does not have access to the signal itself, but to a measurement

Example: neural spikes measured in non invasive manner ;)

\[ f(t) \]

\[ F(\omega) \]
Example: photographing stars

Can we sample such signals that we see through an imperfect measuring device?

There are many parametric signals which are far from bandlimited

Example: CDMA

Note: rate of transition is finite, given by the chip rate symbol rate much slower
Example: Woodcut pictures
2 Signals of Finite Rate of Innovation

What is so special about a signal \( f(t) \) bandlimited to \([-\omega_m, \omega_m]\)?

With a sampling interval of \( T = \pi/\omega_m \) the signal \( f(t) \) is specified by

\[
\rho = 1/T = \omega_m/\pi
\]

degrees of freedom per unit of time. By the interpolation formula (1), any bandlimited signal can be generated as

\[
\sum_{n \in \mathbb{Z}} f(nT) \delta(t - nT) \ast \text{sinc}(t/T) = f(t) = (1)
\]
Signals of finite \( \rho \) (2)

**Definition:** The number of degrees of freedom per unit of time is called the rate of innovation \( \rho \).

**Rate of innovation**

- Assume a class of signals having a parametric representation
- Consider one signal \( x \) from the class
- Call \( C_x(t_0, t_1) \) the number of degrees of freedom in

- Then

\[
\rho = \lim_{\tau \to \infty} \frac{1}{\tau} C_x\left(\frac{-\tau}{2}, \frac{\tau}{2}\right)
\]

- If \( \rho < \infty \), we call \( x \) a signal of **finite rate of innovation**
Example: Poisson process

Interarrival times: i.i.d. , pdf $\mu e^{-\mu t}$

Expected interarrival time: $1/\mu$

$\{t_i\}$ is a sufficient description of a realization

$$\rho = \frac{1}{E(\text{int. time})} = \mu$$
Aquisition Model, Notation

\[ y_n = y(nT) = \langle h(t - nT), x(t) \rangle \]

**where**
- \( x(t) \): signal
- \( h(t) \): sampling kernel
- \( y(t) \): filtered version of \( x(t) \)
- \( y_n \): samples
Natural questions

1. What are interesting classes of signals with finite $\rho$

2. For which of these classes can we find unique representations through sampling (in particular uniformly) that is:

$$y_n = \langle h(t - nT), x(t) \rangle$$

such that $x \Leftrightarrow y_n$

just like in the bandlimited case

3. What are good kernels $h(t)$?

4. What are the algorithms to find $x(t)$ from $y_n$?
1. "Classic", subspace case. Given known fct $\varphi(t)$:

$$\chi(t) = \sum_{n \in \mathbb{Z}} c_n \varphi\left(\frac{t}{T} - n\right)$$

**Space:** $\text{Span} \left\{ \varphi\left(\frac{t}{T} - n\right) \right\}$

This is a well studied case (sampling, non-uniform sampling, reconstruction). It is a linear problem.

**Example:** Bandlimited signals $[-w_m, w_m]$, $\varphi(t) = \text{sinc}(t)$

**Basis:**

**Ex:**

**Example:** Uniform, B-splines,

**Basis:**

**Ex:**
2. Arbitrary shifts, known \( \varphi(t) : x(t) = \sum_{n \in \mathbb{Z}} c_n \varphi \left( \frac{t}{T} - \tau_n \right) \)

This is not a subspace!

Example: Non-uniform splines

\[
x(t)
\]

periodic non-uniform spline (deg. 1)
3. **Arbitrary shifts, set of known fcts \( \varphi_r(t) \):**

\[
x(t) = \sum_{n \in \mathbb{Z}} \sum_{r=0}^{R} c_{nr} \varphi\left(\frac{t - \tau_n}{T}\right)
\]

**Example:** Non-uniform piecewise polynomials
The 4 cases of interest:

1. Periodic signal, infinite kernel

2. Finite signal, infinite kernel

3. Infinite signal, finite kernel

4. Infinite signal, infinite kernel

Note: 1, 2 and 3 lead to finite dimensional problems
3 The periodic case

Fourier series

\[ x(t) = \sum_{m \in \mathbb{Z}} X[m] e^{\frac{j2\pi mt}{\tau}} \]

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3. A Periodic “stream” of Diracs

**K Diracs per τ:**

2*K degrees of freedom $\rho = \frac{2K}{\tau}$

$$x(t) = \sum_{n \in \mathbb{Z}} c_n \delta(t - t_n) = \sum_{n \in \mathbb{Z}} \sum_{k=0}^{K-1} c_k \delta(t - t_k - n\tau) = \sum_{k=0}^{K-1} c_k \frac{1}{\tau} \sum_{m \in \mathbb{Z}} e^{\frac{j2\pi m(t - t_k)}{\tau}}$$

*or* $X[m] = \frac{1}{\tau} \sum_{k=0}^{K-1} c_k e^{\frac{-j2\pi mt_k}{\tau}} \quad m \in \mathbb{Z}$

⚠️ *X[m] is a weighted sum of K exponentials* $\left( e^{\frac{-j2\pi t_k}{\tau}} \right)^m$
Consider:

\[ A(z) = \sum_{m=0}^{K} A[m] z^{-m} = \prod_{k=0}^{K-1} \left(1 - e^{-\frac{j2\pi t_k}{\tau}} \cdot z^{-1}\right) \]

Now, note that

\[
\left[1, -e^{-\frac{j2\pi t_k}{\tau}}\right] \ast \left[\ldots, e^{-\frac{j2\pi t_k}{\tau}}, 1, -e^{-\frac{j2\pi t_k}{\tau}}, e^{-\frac{j4\pi t_k}{\tau}}, \ldots\right]
\]

is zero, from which follows that \( A[m] \ast X[m] = 0 \)

Equivalently, in time domain

\[
a(t) = A(z) \bigg|_{z = e^{-\frac{j2\pi t_k}{\tau}}} = \prod_{k=0}^{K-1} \left(1 - e^{-\frac{j2\pi(t_k-t)}{\tau}} \right)
\]

has zeros at \( t = t_k\), \( k = 0, \ldots, K-1\), thus \( a(t) \cdot x(t) = 0 \)

\( A(z) \) is called an annihilating filter, since it “kills” \( x(t) \)

ECC: error locator polynomial
**Theorem 1**: Consider a periodic stream of $K$ Diracs, of period $\tau$, weights $\{c_k\}$ and locations $\{t_k\}$. Take a sampling kernel

$$h_\beta(t) = \beta \text{sinc}(\beta t)$$

where

$$\text{sinc} = \text{I}[-\pi, \pi]$$

$$\beta = \frac{2K + \frac{1}{2}}{\tau} > \rho$$

Pick $N = 2K + 1$ and $T = \tau/N$. Then

$$y_n = \langle h_\beta(t - nT), x(t) \rangle, \ n = 0, ..., N - 1$$

is a sufficient characterization of $x(t)$. 
**Proof**

1. $y_n$ is a sufficient characterization of $X[m]$, $m = -K \ldots K$

Either use Poisson $\sum A[m]z^{-m}$, or graphically:

2. Finding $A[m]$ s.t. $A[m]^*X[m] = 0$ $A[0] = 1$, solve for $m = 1 \ldots K$. This leads to a Toeplitz system, e.g. $K = 3$

$$
\begin{bmatrix}
X[0] & X[-1] & X[-2] \\
X[1] & X[0] & X[-1] \\
\end{bmatrix}
\begin{bmatrix}
A[1] \\
A[3]
\end{bmatrix}
= -
\begin{bmatrix}
X[1] \\
X[2] \\
X[3]
\end{bmatrix}
$$

**Classic Yule-Walker system**

Unique solution for distinct Dirac locations
3. Factorisation of $A(z)$: $A(z) = \prod_{k=0}^{K-1} (1 - u_k z^{-1})$

where $u_k = e^{-j2\pi \frac{t_k}{\tau}}$, thus $\{t_k\}_{k=0}^{K-1}$ is found

4. Finding the weights $c_k$.

Given $\{t_k\}$, $K$ values of $X[k]$ are given,

for ex. for $K = 3$

$$
\begin{bmatrix}
X[0] \\
X[1] \\
X[2]
\end{bmatrix} = \frac{1}{\tau}
\begin{bmatrix}
1 & 1 & 1 \\
u_0 & u_1 & u_2 \\
2u_0^2 & 2u_1^2 & 2u_2^2
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1 \\
c_2
\end{bmatrix}
$$

which is a Vandermonde system, having always a solution given distinct $t_k$'s.
Interpretation

The projection of $x(t)$ onto the lowpass space $BL\left[-\frac{K2\pi}{\tau}, \frac{K2\pi}{\tau}\right]$ is one-to-one for a periodic stream of $K$ Diracs.

**Corollary 1:** Given $A[m], m = 0...K$ and $X[m], m = -K...K$ one can recover the entire spectrum as

$$X[m] = -\sum_{k=1}^{K} A[k]X[m-k], m = K+1...$$

**Proof:** left to the reader

**Notes:**
1. annihilating filter known in sinusoidal retrieval from noise
2. same filter used in error correction coding, and called error locator polyn
3. recursive spectrum extrapolation known as Berlekamp-Massey algo. in ECC 2,3 over finite fields...
3.B Non-uniform splines

A signal \( x(t) \) is a periodic non-uniform spline of degree \( R \) with \( K \) knots at \( \{ t_k \}_{k=0}^{K-1} \) iff its \( (R + 1)^{th} \) derivative is periodic of the form

\[
x^{(R+1)}(t) = \sum_{m \in \mathbb{Z}} c_m \delta(t - t_m)
\]

where \( t_{m+k} = t_m + \tau \)

Clearly, the Fourier series satisfy

\[
X^{(R+1)}[m] = \left( \frac{j2\pi m}{\tau} \right)^{R+1} X[m] \quad (\ast)
\]

Thus
**Theorem 2:** Consider a periodic non-uniform spline of max degree $R$ and period $\tau$. Take $h_\beta(t)$ as sampling kernel, with

$$\beta = \frac{2K + 1}{\tau} \quad \text{and} \quad T = \frac{\tau}{N} \quad N = 2K + 1$$

Then

$$y_n = \langle h_\beta(t - nT), x(t) \rangle \quad n = 0 \ldots N - 1$$

uniquely defines $x(t)$.

**Proof:** similar to Thm 1 to get $X[m]$. Then $X^{(R+1)}[m]$ follows from (*), to which we apply Thm 1. $X[0]$ is added at the end.
3.C Derivatives of Diracs

\[ \delta^{(r)}(t) : \int f(t)\delta^{(r)}(t - t_0)dt = (-1)^{r}f^{(r)}(t_0) \]

where \( f \) is \( r \)-times differentiable

Then a periodic stream of differentiated Diracs is

\[ x(t) = \sum_{m \in \mathbb{Z}} \sum_{r=0}^{R_{m-1}} c_{mr} \delta^{(r)}(t - t_m) \]

There are: \( K \) locations, \( \tilde{K} = \sum_{k=0}^{K-1} R_k \) weights.

Thus: \( \rho = \frac{K + \tilde{K}}{\tau} \)

It can be verified that:

\[ X[m] = \frac{1}{\tau} \sum_{k=0}^{K-1} \sum_{r=0}^{R_{m-1}} c_{kr} \left( \frac{j2\pi m}{\tau} \right)^r e^{-j2\pi mt_k} \]
The annihilating filter now requires multiples zeros, since

\((1-u_kz^{-1})^R\) annihilates \(m^{R-1}u_m^k\). Thus \(A(z)\) becomes

\[
A(z) = \prod_{k=0}^{K-1} (1-u_kz^{-1})^{R_k}
\]

Then: \(A[m]^*X[m] = 0\), therefore, one can show:
**Theorem 3:** Consider a periodic stream of differentiated Diracs as above. Take as sampling kernel \( h_\beta(t) = \beta \text{sinc}(\beta t) \) with \( \beta = \rho + 1/\tau \) and sample \( h_\beta Sx \) at \( N \) points \( t = n\tau/N \) where \( n = 0 \ldots N-1 \) and \( N = K+K+1 \). Then

\[
y_n = \langle h_\beta(t - n\frac{\tau}{N}), x(t) \rangle \quad n = 0 \ldots N-1
\]

is a sufficient characterization of \( x(t) \).

**Proof:** Similarly to Thm 1, we first get \( X[m] \) from \( y_n \). Then we solve for the location \( \{t_k\} \ A[m]X[m] = 0 \) and finally for the coefficients \( \{c_{kr}\} \). The latter calls for a generalized Vandermonde system which is non-singular for \( t_i \neq t_j \quad i \neq j \).
3.D Piecewise Polynomials

A periodic piecewise polynomial $x(t)$ with $K$ pieces of degree $\max R$ has an $(R+1)^{th}$ derivative which is a stream of differentiated Diracs, or

$$x(t) = \sum_{m \in \mathbb{Z}} \sum_{r=0}^{R_m-1} c_{mr} \delta^{(r)}(t-t_m)$$

There are: $K$ locations, $\tilde{K} = (R+1)K$ weights

$$\rho = \frac{(R+2)K}{\tau}$$
Then:

**Theorem 4:** A signal defined by its derivatives as in (***) can be recovered after convolution by $h_\beta(t)$, where $\beta = \rho + 1/\tau$ and sampling at $t = n\tau/N$ with $N = (R + 2)K + 1$, that is

$$y_n = \langle h_\beta(t - n\frac{\tau}{N}), x(t) \rangle \quad n = 0 \ldots N - 1$$

uniquely specifies $x(t)$

**Proof:** left to the reader, along Theorem 1, 2 and 3.
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\[ x(t) \]

\[ h(t) \]

\[ y(t), y_n \]

**Time**

**Freq**
4 Finite Length Signals

A finite length signal with finite \( \rho \) clearly has a finite # of degrees of freedom.

The question of interest is:

given a sampling kernel with an infinite support (like the sinc or the gaussian), is there a finite set of samples that uniquely specifies the signal?
4.1 Gaussian Kernel

Consider the same signal as in (4.1), now using a gaussian kernel

\[ h(t) = e^{-\frac{t^2}{2\sigma^2}} \]

Then, the sample values are

\[ y_n = \langle x(t), e^{-\left(\frac{t}{T} - n\right)^2 / (2\sigma^2)} \rangle \]

\[ y_n = \sum_{k=0}^{K-1} c_k e^{-\left(\frac{t_k}{T} - n\right)^2 / (2\sigma^2)} \]
Expanding (4.1)

\[ y_n = \sum_k c_k e^{T^2 \sigma^2} \cdot e^{T^2 \sigma^2} \cdot e^{2\sigma^2} \]  

(4.2)

Introduce

\[ Y_n = e^{T^2 \sigma^2} \cdot y_n \]

Thus

\[ Y_n = \sum_k \frac{-t_k^2}{a_k} \cdot \left( \frac{t_k}{e^{T^2 \sigma^2}} \right)^n \cdot \left( \frac{t_k}{a_k} u_k^n \right) \]

\[ Y_n = \sum_{k=0}^{K-1} a_k u_k^n \]  

(4.4)

that is ... a linear combination of exponentials!
Therefore, use the usual method of the good old annihilating filter

\[ \text{A} \star \text{Y} = 0 \]

and factor it such as to find \( \{u_k\}_k = 0...K - 1 \)

From \( u_k \):

\[ t_k = 2\sigma^2 T \ln u_k \]

From \( u_k \) and \( t_k \) and \( K \) values of \( Y_n \), we can solve for \( c_k \) in (4.2). Thus
Theorem 5: Given a finite stream of \( K \) Diracs and a gaussian kernel \( h(t) = e^{-t^2/(2\sigma^2)} \), then \( N \) samples

\[
y_n = \langle x(t), h\left(\frac{t}{T} - n\right) \rangle
\]

where \( N \geq 2K \), are sufficient to reconstruct the signal.

Note: Similar remarks as for Theorem 3...

But: Here, unlike in the sinc case, we have an "almost local" reconstruction because of the exponential decay of \( h(t) \) !
4.2 Sinc kernel (Thierry’s tour de force)

Consider a finite sequence of spikes

\[
x(t) = \sum_{k=0}^{K-1} C_k \delta(t - t_k)
\]

and a kernel \( \text{sinc}(t/T) \)

The samples \( y_n = \langle x(t), \text{sinc}\left(\frac{t}{T} - n\right) \rangle \) are

\[
y_n = \sum_{k=0}^{K-1} C_k \text{sinc}\left(\frac{t_k}{T} - n\right) = (-1)^n \sum_{k=0}^{K-1} C_k \frac{\sin(\pi t_k / T)}{\pi \left(\frac{t_k}{T} - n\right)}
\]
Introduce the following interpolators:

\[ P(u) = \prod_{k=0}^{K-1} \left( \frac{t_k}{T} - u \right) = \sum_{k=0}^{K} p_k u^k, \text{ deg. } K \]

\[ P_l(u) = \prod_{k \neq l} \left( \frac{t_k}{T} - u \right), \text{ deg. } K - 1 \]

Then, consider the following

\[ Y_n = (-1)^n P(n) y(n) = \frac{1}{\pi} \sum_{k=0}^{K-1} C_k \sin((\pi t_k)/T) P_k(n) \quad (4.7) \]

\[ Y = A \cdot C \]

Now (key insight!) \( Y_n \) is of degree \( K - 1 \) Thus

\[ \Delta^K Y_n = 0 \quad n = K \ldots N - 1 \quad (4.8) \]

\[ V \cdot p = 0 \quad N - K \geq K \]

Note: \( \sim \Delta^K \) similar to annihilating filter
So, as long as $N - K \geq K$, one can use (4.4) to solve for $P_k$ from $y_n$. This leads to $\{t_0, t_1, ..., t_{K-1}\}$.

Using this in (4.6) allows to solve for $\{c_i\}$. Thus:

**Theorem 6:** Given a finite stream of $K$ Diracs and a $\text{sinc}(t/T)$ kernel, $N$ samples $y_n = \langle x(t), \text{sinc}\left(\frac{t}{T} - n\right) \rangle_{n=0...N-1}$ where $N \geq 2K$, are sufficient to reconstruct the signal.

Note: the result does not depend on $T$! Of course, it shows up in the conditionning of linear system!!
The steps to reconstruct the signal are

1. Solve a linear system $K \times K$

   $$\{y_i\} \rightarrow \{p_i\}, \ i = 0\ldots K-1 \ (p_k = 1)$$

2. Factor

   $$P(u) \rightarrow \{t_i\}, \ i = 0\ldots K-1$$

3. Solve linear system $\rightarrow \{c_i\}$

This method can be extended to piecewise polynomials, similarly to Theorem 4.

Also, there is an obvious equivalent for discrete-time signals from $l_2(\mathbb{Z})$ and discrete-time sinc kernels.
Sinc Kernel, finite length signals

Conditioning on location

Conditioning on weights
5. Applications

We show 2 direct applications of the results shown above.

5.1 Piecewise Bandlimited Signals

Consider a signal that is the sum

\[ x = x_{BL} + x_{PP} \]

where \( x_{BL} \) is bandlimited and \( x_{PP} \) is piecewise polynomial.

Assume \( x_{BL} \) is specified by its frequency component \( X_{BL}[k], k \in [-M, M] \) while \( x_{PP} \) has \( 2K \) degrees of freedom.
Then, consider the spectrum of $X[k]$, $k \in [-M-2K, M+2K]$.

First, using $X[k]$, $k \in [M+1, M+2K]$ and the technique of Proposition 1 or Theorem 1, we can recover $x_{PP}$. Subtracting $X_{PP}$ from $X$, we can then recover $X_{BL}$. 
Piecewise Bandlimited Signal
Thus:

**Proposition 3**: Given a piecewise BL signal of length $N$, with $2M + 2K$ degrees of freedom. Pick $Q$ a divisor of $N$ and $\varphi[n] = \text{IDTFS}(I[-2K-M,M+2K])$.

Then

$$y[l] = \langle x[n], \varphi[n-lQ]\rangle_{\text{circ}}$$

uniquely specify $x[n]$ if

$$\frac{N}{2Q} > M + 2K$$

**The proof follows from earlier results with adjustments ●**
5.2 Filtered Piecewise Polynomials

Consider a stream of \( K \) Diracs convolved with a known filter \( g(t) \)

Thus: \( x(t) = g(t) \ast d(t) \)

where \( g \) is known and \( d(t) = \sum_i \alpha_i \delta(t - t_i) \)

Clearly, if \( g[n] \leftrightarrow G[k] \) is invertible over \( 2K \) frequency values, then we can use Proposition 1.
Example:
In particular:

Proposition 4: Assume $x[n]$ with $K$ Diracs and a filter $G[k] \neq 0$, $k \in [-K, K]$. The signal we observe is $x[n] * g[n]$.

Using $\varphi[n] = \text{IDTFS}(I_{[-K, K]})$ and $M$ such that $\frac{N}{2M} > K$, $M$ a divisor of $N$.

Then

$$y[l] = \langle x[n], \varphi[n - lM] \rangle$$

is a sufficient representation of $x[n]$.

A more difficult case appears when $g[n]$ is unknown but of finite $\rho$ ...
6 Multidimensional Case

2D Poisson: K Diracs on $\mathbb{R}^2/T$

Various approaches

• non separability is the key!
• $X[m_1, m_2], |m_i| \leq K$ is sufficient $\Rightarrow O(K^2)$ samples
• $X[m_1, m_1], |m_1| \leq K$ is sufficient $\Rightarrow O(K)$ samples

--> 2D root finding (...) or spectral extrapolation

Extension:

• lines
• simples objects

Goal: $\#\text{samples} \sim \#\text{deg. of freedom of object}$
Example of a 2D gaussian kernel:
2D methods based on projections

Radon Transform

\[ f(x, y) \Leftrightarrow F(\theta, t) \]

What about “finite complexity” objects?
⇒ Projections are finite rate of innovation!
Result: Set of $K$ Diracs can be perfectly reconstructed from $K+1$ bandlimited projections with $2K$ samples

See [Maravic] ICASSP-2002
Many com. systems use wideband signalling

CDMA: chip rate $>>$ symbol rate

UWB: pulse position modulation

In both cases

rate of innovation $<<$ bandwidth

But: Noise!

Solution: oversample

subspace methods, SVD
7.1 Solving for sinusoids in noise

Idea: Solve for “longer” filter:

\[
\begin{bmatrix}
  x(0) & x(-1) & \cdots & x(-M) \\
  x(1) & x(0) & & \\
  x(2) & & \cdots & \\
  \vdots & & \ddots & \\
  x(M) & \cdots & x(0)
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_{M-1}
\end{bmatrix}
= \begin{bmatrix} x(1) \\
\end{bmatrix}
\]

using 2M+1 samples > 2K oversample

Now: The noiseless Toeplitz matrix has rank K (# of sinusoids) with

\[
A = \begin{bmatrix} a_0 & a_1 & \cdots & a_{K-1} \end{bmatrix}
\]

where \( a_i = \begin{bmatrix} e^{-j\omega_i M} & \cdots & 1 & \cdots & e^{j\omega_i M} \end{bmatrix}^T \)
we can write the Toeplitz matrix as

\[
T = A \cdot \begin{bmatrix}
\alpha_0 \\
\alpha_1 \\
\vdots \\
\alpha_{K-1}
\end{bmatrix} \cdot A^M + N
\]

where \( N \) is the noise Toeplitz matrix

Thus: If the sinusoids dominate the noise (\( M \) large enough), a \( K \)-dimensional subspace identifies the sinusoids

Then:

1. Compute SVD of \( T \)
2. Approximate by \( K \) largest singular value: \( T \rightarrow \hat{T} \)
3. Solve \( \hat{T}a = x \) on subspace
4. Find roots closest to U.C.

\[\Rightarrow \text{best approximation of sinusoids}\]
Note:

- Many alternative available
- Well studied problem
- Time versus correlation domain

Example: - MUSIC
          - ESPRIT
          - NL
7.2 Multiuser Communication

Direct Sequence Code Division Mult. Access (DS-CDMA)

Model:
- User i has a signature sequence $S_i$
- each bit is spread into this signature

Ex:

Thus:

Clearly: rate of innovation is symbol rate

Usually: sampling done at chip rate or faster

Now: chip rate $10^2 - 10^3 >$ symbol rate! (e.g. $L=511$)
But:  - multiaccess scheme 
    - multipath environment

Multiaccess: signature are orthogonal 
Multipath: small number of dominant pulses

User i: \( p_i(t) = \sum_{k=1}^{p} \beta_i \delta(t - t_k^{(l)}) \)

Two phases
1. Channel estimation:
   Using training sequences, \( \{p_i(l)\}_{i=1..K} \) is estimated

2. Detection:
   Based on the channel estimate, various detectors (e.g. MMSE) can be applied

Question: For a digital receiver,
Should one run:
   - channel estimation
   - detection
at symbol rate or chip rate?
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Detect.1

Detect.K

Chann. Est.

K Users

Signatures
$S_i \perp S_j$

Channels

MAC

Chann. Est.

Detection

analog
Degrees of freedom

Channels:
- $K$ users
- $P$ multiple paths

But: users can use training sequences of length $K$

Result:
Solving $K$ linear systems of $O(M)$ with $M \geq 2P$, is sufficient for channel estimation
7.3 Ultrawideband communications

Very low signal to noise ratio (-15 dB)

Used for communications in unlicensed spectrum and for ranging applications

Bandwidth: several GHz
Very difficult to design digital receivers

Results:
Finding one dominant eigenvalue can be sufficient!
8 Conclusions

We have seen:

- Many signals that look “unsampleable” actually can be sampled at their rate of innovation!
- Methods: give me an exponential and I will annihilate it!
- Structured linear systems with fast algorithms $O(K^2)$
- Can be generalized (rotational, 2D)

But: There are many more signals with finite rate of innovation

Conjecture: They can be sampled at or above their rate of innovation!
Outlook

• Many other parametric classes are of interest (piecewise trigonom.)

• Often, there is a “low degree of freedom” explanation

• This is not necessarily a subspace (e.g. manifold)

• “Super-resolution” signal processing for appropriate models (channels, images, etc...) has great potential

Occam’s Razor for sampling!
References


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