Homework 4
Thursday, October 22, 2015

Exercise 1. Norms of Oblique Projections

Orthogonal projections are contractions—they either make a vector shorter, or they don’t change its length. If \( \Pi : \mathcal{H} \rightarrow \mathcal{H} \) is an orthogonal projection in a Hilbert space \( \mathcal{H} \), we can write this as
\[
\|\Pi x\| \leq \|x\|, \quad \forall x \in \mathcal{H}.
\]
Equivalently, we can say that the spectral norm of \( \Pi \) is upper bounded by 1, and in fact it is equal to one.

(i) Show that \( I - \Pi \) is also an orthogonal projection, and that we have
\[
\|\Pi\| = \|I - \Pi\| = 1.
\]
Assume that the range of \( \Pi \) is non-trivial, and that it is not the whole space \( \mathcal{H} \).

For non-orthogonal projections, it no longer holds that they are restrictions—they can make a vector longer as well. Thus for a general projection \( P : \mathcal{H} \rightarrow \mathcal{H} \), we will have
\[
\|P\| \geq 1.
\]
However, it still holds that \( \|P\| = \|I - P\| \), and it is the goal of the rest of this exercise to prove this.

Let \( P \) be a projection (not necessarily orthogonal), such that neither \( \mathcal{R}(P) \) nor \( \mathcal{N}(P) \) equal \( \mathcal{H} \). Let \( u \in \mathcal{H} \) be such that \( \|u\| = 1 \). Let \( x = Pu \) and \( y = (I - P)u \).

(ii) Show that \( \|u\|^2 = \|x\|^2 + \|y\|^2 + 2 \text{Re} \left( \langle x, y \rangle \right) \).

(iii) We want to show that \( \|Pu\| \leq \|I - P\| \). Show first that this holds in special cases when \( x = 0 \) or \( y = 0 \) (the general claim when \( x \neq 0 \) and \( y \neq 0 \) is the subject of the next two subquestions).

\( \text{Hint: Recall that the norm of any projection is greater than or equal to 1.} \)

(iv) Assume that \( x \neq 0 \) and \( y \neq 0 \). Define \( w \in \mathcal{H} \) as \( w = \tilde{x} + \tilde{y} \), with
\[
\tilde{x} = \frac{\|y\|}{\|x\|} x, \quad \tilde{y} = \frac{\|x\|}{\|y\|} y.
\]
Prove that \( \|w\| = 1 \).

(v) Show that \( \|Pu\| = \|(I - P)w\| \), and then use the definition of the spectral norm to prove that \( \|Pu\| \leq \|I - P\| \). Use it again to prove \( \|P\| \leq \|I - P\| \).

(vi) Use symmetry to show that \( \|I - P\| \leq \|P\| \). Conclude finally that \( \|I - P\| = \|P\| \).
Exercise 2. The Filtering Operator
Consider the analog domain filtering operator defined as a linear mapping $A : \mathcal{L}^2(\mathbb{R}) \rightarrow \mathcal{L}^2(\mathbb{R})$ given by

$$Af = g$$

where $g(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau, \quad t \in \mathbb{R}.$

Let $H(\omega)$ denote the Fourier transform of the impulse response function $h(t)$. Assume that $H(\omega)$ is absolutely bounded with $\sup_{\omega \in \mathbb{R}} |H(\omega)| = \overline{H} < \infty$.

(i) Show that the operator norm of the operator $A$ satisfies $\|A\| \leq \overline{H}$.

Recollect that operator norm is defined as

$$\|A\| = \sup_{x \in \mathcal{L}^2(\mathbb{R}) : \|x\| = 1} \|Ax\|.$$

Hint: you can use Parseval’s equality

(ii) Identify the adjoint operator $A^*$.

(iii) What properties must be satisfied by the Fourier transform $H(\omega)$ to ensure that $A$ is

(a) a self-adjoint operator?

(b) an orthogonal projection operator?

Justify your answers.

(iv) If there exists $\omega_0 \in \mathbb{R}$ such that $H(\omega)$ is continuous at $\omega_0$ and $|H(\omega_0)| = \overline{H}$, show that $\|A\| = \overline{H}$.

Exercise 3. Fourier Series and Fourier Transform
Let $x$ be a triangle wave with period $T = 1$ as in Figure 1, whose values within a single period are given by

$$x(t) = |t|, \quad t \in \left[-\frac{1}{2}, \frac{1}{2}\right).$$

![Figure 1: The triangle wave.](image)

(i) Find the Fourier series expansion of $x(t)$.

(ii) Let the signal $y(t)$ be

$$y(t) = \begin{cases} x(t) & t \in \left[-\frac{1}{2}, \frac{1}{2}\right), \\ 0 & \text{otherwise}. \end{cases}$$

Compute the Fourier transform of $y(t)$.

(iii) Based on the result in (ii), compute the Fourier transform of $x(t)$. What can you say about the relationship between the Fourier transform $X(\omega)$ and its Fourier series coefficients $X_k$ of the periodic function $x(t)$?

Hint: you may need to use the Fourier transform of a Dirac-comb

$$\sum_{n \in \mathbb{Z}} \delta(t - nT) \xrightarrow{F} \frac{2\pi}{T} \sum_{k \in \mathbb{Z}} \delta(\omega - \frac{2\pi}{T} k).$$
(iv) Based on the result in (i), show that
\[
\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.
\]

**Exercise 4. Fourier Transforms**

(i) Given the function \( f(t) = \text{sinc}^2(\frac{\pi t}{2}) = \frac{\sin^2(\pi t/2)}{(\pi t/2)^2} \), find its Fourier transform \( F(\omega) \), and sketch plots of \( f(t) \) and \( |F(\omega)| \). On each plot, indicate the relevant points, such as the value at \( t = 0 \) and \( \omega = 0 \), and the values where function has zeros.

(ii) Let \( g(t) = f(t) \cos(2\pi t) \). Find \( G(\omega) \) and sketch the plot of \( |G(\omega)| \).

(iii) Let \( x(t) = (f * g)(t) \). Give the values \( x_{\text{min}} = \inf_{t \in \mathbb{R}} x(t) \) and \( x_{\text{max}} = \sup_{t \in \mathbb{R}} x(t) \). Explain your answer.

![Figure 2: Magnitude of the Fourier transform \( U(\omega) \) of \( u(t) \).](image)

(iv) Let an unknown, non-negative real function \( u(t) \) have a Fourier transform \( U(\omega) \) whose magnitude is given in Figure 2. Let \( y(t) = (f * u)(t) \). Give the upper bound on \( y_{\text{max}} = \sup_{t \in \mathbb{R}} y(t) \), and a particular signal \( u(t) \) for which that bound is reached. Explain your answer.

*Hint: You can use the Cauchy-Schwarz inequality.*

(v) Using the Poisson summation formula and the signal \( g(t) \), show that
\[
\sum_{n=-\infty}^{\infty} \frac{1}{(n + t)^2} = \frac{\pi^2}{\sin^2(\pi t)}, \quad \text{for all } t \in \mathbb{R} \setminus \mathbb{Z}.
\]

*Hint: Do not be confused if you manage to show "only" for \( t' = \frac{1}{2} \); the equality still holds by mere variable renaming.*
Exercise 5. Mutirate identities
In parts (i)-(iv) of this exercise, you should express the $z$-transform $Y(z)$ of the sequence $y_n$ in terms of the $z$-transform $X(z)$ of the input sequence $x_n$. Note that in part (iv), $Y(z)$ is also a function of the delay $N$, which you should explicitly express.

Exercise 6. MATLAB Exercise: Multirate Systems
download and load the file hw4.mat. This file contains an image $X$ and a one dimensional filter $c$.

(i) Draw the magnitude of Fourier transform of filter $c$ using MATLAB (Do not forget to use \texttt{fftshift} function). What kind of filter is $c$?

(ii) Construct a filtering circulant matrix $C$ using the filter $c$.

(iii) Construct a matrix $D_2$ which acts as a downsampler by 2. Apply it to $X$ (use the command $D_2*X*D_2'$) and check the result (you can use the function \texttt{imshow()}). Why do you see strange shapes appearing in the image?

(iv) Using $C$ and $D_2$, construct a decimation matrix and call it $Cd$. Apply it to the image $X$ and call the output $Y$. Compare the result with the the result of part [iii] Which one is more appealing?

(v) Construct a matrix $U_2$ which acts as an upsampler by 2. Apply it to $Y$ and show the result.

(vi) Using $C$ and $U_2$, construct an interpolation matrix and call it $Cu$. Apply it on the image $Y$ and compare the result with the output of part [v] Which one is more appealing?