Exercise 1. Sampling and interpolation

(i) Let $\tilde{\Phi}^* : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ be a finite-dimensional sampling operator represented in matrix form as

$$
\tilde{\Phi}^* = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}
$$

(a) What is $\mathcal{N}(\tilde{\Phi}^*)^\perp$ where $\mathcal{N}$ stands for the null space?
(b) What is the interpolation operator $\Phi$ that is ideally matched to $\tilde{\Phi}^*$?
(c) Given a vector $x \in \mathbb{R}^4$, obtain a formula for $\hat{x} \in \mathbb{R}^4$ such that $\hat{x}$ is of the form

$$
\hat{x} = \begin{bmatrix}
a \\
a \\
b \\
b
\end{bmatrix}
$$

for some $a, b \in \mathbb{R}$ and $\|x - \hat{x}\|$ is minimized subject to this constraint. *Hint: Use the result from the previous question.*
(d) Given a vector $x \in \mathbb{R}^4$, obtain a formula for $\hat{x} \in \mathbb{R}^4$ such that $\hat{x}$ is of the form

$$
\hat{x} = \begin{bmatrix}
c \\
c + d \\
c + d \\
d
\end{bmatrix}
$$

for some $c, d \in \mathbb{R}$ and $\|x - \hat{x}\|$ is minimized subject to this constraint. *Hint: Follow the same steps as above for an alternative choice of the sampling operator $\tilde{\Phi}^*$.*

(ii) Consider the sampling operator $\tilde{\Phi}^* : L^2(\mathbb{R}) \rightarrow \ell^2(\mathbb{Z})$ depicted in Figure 1(a) with $T = 1$. Here $x(t)$ is a signal in $L^2(\mathbb{R})$ and $\tilde{g}(t)$ is the box-filter with impulse response given by

$$
\tilde{g}(t) = \begin{cases}
1, & \text{if } |t| < \frac{1}{2} \\
0, & \text{otherwise}
\end{cases}
$$

Figure 1: Sampling and interpolation operators
(a) What is the interpolation operator \( \Phi \) of the form in Figure 1(b) that is ideally matched to \( \tilde{\Phi}^* \)?

(b) Given a signal \( x(t) \in \mathcal{L}^2(\mathbb{R}) \) identify a piecewise-constant signal \( \hat{x}(t) \) such that \( \| x - \hat{x} \| \) is minimized subject to the constraint that for every integer \( n \) the signal \( \hat{x}(t) \) is equal to a constant in the interval \([n, n+1)\). Justify your answer.

Exercise 2. Fourier Series, Fourier Transform

Consider a periodic stream of Diracs with period \( \tau = 1 \):

\[
x(t) = \sum_{k=1}^{K} x_k \delta(t - t_k), \quad t \in [0, 1),
\]

where \( \delta(\cdot) \) is the Dirac delta function.

(i) Verify that the Fourier series coefficients \( \hat{x}_m \) of the periodic signal \( x(t) \) are

\[
\hat{x}_m = \sum_{k=1}^{K} x_k e^{-j2\pi mt_k}.
\]

(ii) Suppose the \( z \)-transform of a filter \( \{h_k\}_{k=0}^{K} \) is given as

\[
H(z) = \sum_{k=0}^{K} h_k z^{-k} = \prod_{k=1}^{K} \left(1 - e^{-j2\pi t_k} z^{-1}\right).
\]

Show that the convolution between the discrete filter coefficients \( \{h_k\}_{k=0}^{K} \) and the Fourier series coefficients \( \hat{x} \) is zero:

\[
[h \ast \hat{x}]_m = 0 \quad \forall m.
\]

Here \( [\cdot]_m \) means the \( m \)-th component of the convolution result.

(iii) Suppose the periodic Dirac stream \( x(t) \) is first convolved with a sinc window of bandwidth \( B = 5 \) and uniformly sampled with sampling step \( T = 1/N \), where \( N \) is the total number of samples taken per period:

Here \( \varphi(t) = \text{sinc}(5\pi t) \). Express the ideally lowpass filtered samples \( y_n \)

\[
y_n = \langle x(t), \text{sinc}(5\pi(nT - t)) \rangle_t
\]

in terms of the Fourier series coefficients \( \hat{x}_m \).

Hint: you may want to replace \( x(t) \) with its Fourier series representation. Also, you might need to use 1) the time shifting property of the Fourier transform; and 2) the Fourier transform of a sinc function:

\[
\text{sinc}(5\pi t) \xrightarrow{\text{FT}} \begin{cases} 
\frac{1}{5} & \text{if } |\omega| \leq 5\pi \\
0 & \text{otherwise}.
\end{cases}
\]

(iv) Further, if we assume we have sufficiently many samples \( \{y_n\}_{n=1}^{N} \) with \( N \geq 5 \). How can we compute the Fourier series coefficients \( \hat{x}_m \) from the set of samples \( y_n \)?

Hint: You are supposed to express \( \hat{x}_m \) in terms of \( y_n \) in this part. This is not the same question asked in (iii).
Exercise 3. A Sampling Problem
Assume a sampling setup as follows

where $S$ is a continuous-time system (operator) defined as

$$(Sx)(t) = \int_{t-T/2}^{t+T/2} x(s)ds.$$  

Note that the above system describe a practical sampling and hold system commonly implemented in ADCs.

(i) Give a complete analytic expression for $y_n$

$$y_n = \int (x * h)(t)dt.$$  

(ii) Let $h(t)$ be such that $y(t) = (x * h)(t)$. Find and sketch $H(\omega)$.

(iii) Give the necessary condition on $B_0$ and $T$ such that for every $x(t) \in \{\text{functions band-limited to } [-B_0/2, B_0/2]\}$ we have $\tilde{x}(t) = x(t)$ (this is independent of the choice of $\Phi$).

(iv) Assuming that the necessary condition from 3) is met, what should $\Phi$ be so that $\tilde{x}(t) = x(t)$ for every $x(t) \in \{\text{functions band-limited to } [-B_0/2, B_0/2]\}$? (that is what is the sufficient condition).

(v) If the conditions in 3) and 4) hold, show that the operator

$$P : L^2(\mathbb{R}) \to \{\text{functions band-limited to } [-B_0/2, B_0/2]\}, \quad x \mapsto Px = \tilde{x},$$

is an orthogonal projection.

(vi) What should $\Phi$ be so that $\tilde{Q}$ is the adjoint of $Q$? With such a choice of $\Phi$, is $P$ an orthogonal projection?
Exercise 4. Different sampling schemes

Through this exercise we will explore different sampling setups and their effects on the reconstructed signal using MATLAB. For this purpose, consider the band-limited signal $x(t)$ provided in the file `bl_signal.mat` and the reconstruction scheme of Figure 2 where the interpolation filter is a box function of the form $g(t) = \{ \frac{1}{\sqrt{T}} |t| < T/2 \quad 0 \text{ otherwise} \}$. 

(i) Consider first sampling without pre-filter (i.e., $\tilde{g}(t) = \sqrt{T} \delta(t)$) and no digital filter ($q_n = \delta_n$). Implement the reconstruction chain and numerically compute the reconstructed signal $\hat{x}(t)$ and the reconstruction error $e = \|x(t) - \hat{x}(t)\|_2^2$ for $T = 1, 2, 3$. Plot the spectrum of the sampled signal for $T = 1$ and $T = 3$ and observe the aliasing effect.

(ii) Consider using the box function as a pre-filter $\tilde{g}(t) = g^*(-t)$ and $q_n = \delta_n$. Reconstruct the signal for $T = 1, 2, 3$ and compute the reconstruction error in each case.

(iii) Replace the interpolation filter $g(t)$ by a triangle wave of the form

$$g(t) = \begin{cases} 
1 - |t/T| & -T \leq t < T \\
0 & \text{otherwise}
\end{cases}$$

and consider the use of a digital filter $q_n$ after sampling. Find (analytically) the filter that makes the sampling and interpolation operators consistent. Reconstruct the signal and compute the reconstruction error for $T = 1, 2, 3$.

(iv) For $T = 1$ display in the same plot the reconstructed signal for each sampling setup together with the original signal. Which scheme achieves the best reconstruction? Comment on the effect of the sampling period in the reconstruction error.