Exercise 1. Parallelogram Identity (Section 2.2.3)

(i) Prove the parallelogram identity

\[ \|x - y\|^2 + \|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2 \]

in \( \mathbb{R}^2 \) with norm of \( x = (x_1, x_2) \) defined as \( \|x\|_2 = \sqrt{x_1^2 + x_2^2} \).

(ii) Prove the identity in general, for any norm induced by an inner product.

(iii) Show by a counterexample that when a norm is not induced by an inner product, the identity does not hold.

(iv) Show that when the identity holds for a normed vector space, the norm arises from an inner product defined by \( \langle x, y \rangle = \frac{\|x + y\|^2 - \|x - y\|^2}{4} \).

Exercise 2. Norm Induced by an Inner Product (Section 2.2.3)

Given the following matrix

\[ A = \begin{bmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{bmatrix} \]

for which values of \( a \in \mathbb{R} \) is \( \sqrt{x^*Ax} \) a norm defined on \( \mathbb{R}^3 \)?

Exercise 3. \( p < 1 \) Pseudonorms (Section 2.2.3)

The \( \ell^p \)-norm definition \( \|x\|_p = \left( \sum_{k \in \mathbb{Z}} |x_k|^p \right)^{1/p} \) does not yield a valid norm when \( p < 1 \). It can nevertheless be a useful quantity.

(i) Show that the triangle inequality fails to hold for \( \ell^p \). (It suffices to come up with a single example.)

(ii) Show that for \( x \in \mathbb{R}^N \), \( \lim_{p \to 0} \|x\|_p \) gives the number of nonzero components in \( x \).

(Inspired by—but abusing—this result, it is now common to use \( \|x\|_0 \) to denote the number of nonzero components in \( x \).)

Exercise 4. Distances not Necessarily Induced by Norms (Section 2.2.3)

A distance or metric \( d : V \times V \to \mathbb{R} \) is a function with the following properties:

(i) **Nonnegativity:** \( d(x, y) \geq 0 \) for every \( x, y \) in \( V \).

(ii) **Symmetry:** \( d(x, y) = d(y, x) \) for every \( x, y \) in \( V \).

(iii) **Triangle Inequality:** \( d(x, y) + d(y, z) \geq d(x, z) \) for every \( x, y, z \) in \( V \).

(iv) \( d(x, x) = 0 \) and \( d(x, y) = 0 \) implies \( x = y \).
The discrete metric is given by

\[ d(x, y) = \begin{cases} 
0 & \text{if } x = y, \\
1 & \text{if } x \neq y.
\end{cases} \]

Show that the discrete metric is a valid distance and show that it is not induced by any norm.

Exercise 5. A Vector Space of Functions (Section 2.2.2)

Consider the functions \( \varphi_k(t) = A \text{sinc}(\pi(t - k)) \) where \( \text{sinc}(t) := \frac{\sin(t)}{t} \) and \( k \) is an integer. (You will see by the end of this exercise that \( \varphi_k(t) \in L^2(\mathbb{R}) \)).

(i) For integers \( k, \ell \) evaluate

\[ \int_{\mathbb{R}} \varphi_k(t)\varphi_\ell^*(t)dt. \]

Conclude that \( \varphi_k(t) \in L^2(\mathbb{R}) \), and that \( \{ \varphi_k : k \in \mathbb{Z} \} \) forms an orthogonal set of functions in \( L^2(\mathbb{R}) \).

\text{Hint I: } \int_{\mathbb{R}} \varphi_k(t)\varphi_\ell^*(t)dt = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\varphi}_k(\omega)\hat{\varphi}_\ell^*(\omega)d\omega.

\text{Hint II: } \mathcal{F}\{\text{sinc}(\pi t)\} = \text{rect}(\frac{\omega}{2\pi}).

(ii) For what value(s) of \( A \) do \( \{ \varphi_k : k \in \mathbb{Z} \} \) form an orthonormal set of functions in \( L^2(\mathbb{R}) \)?