**Homework 3 (graded)**

Thursday, October 6, 2016. Due Thursday, October 20, 2016

All exercises carry 20 points. Points from all exercises are summed. In order to get the full grade for this homework, you need 100 points or more.

**Exercise 1. Ranges, nullspaces, sparse vectors**

Consider a matrix $A \in \mathbb{R}^{m \times n}$, with $m < n$.

(i) Show that the mapping $A : \mathbb{R}^n \to \mathbb{R}^m$ is not invertible (in particular, it is not injective).

(ii) Write out a general solution of the equation $y = Ax$.

(iii) Imagine now that we restrict the domain only to vectors in $\mathbb{R}^n$ with at most $k$ non-zero elements. Denote the set of all such vectors by $S_k$. Prove that $A : S_k \to \mathbb{R}^m$ is injective if and only if every subset of $2k$ or less columns of $A$ is linearly independent. Discuss the implications of this result.

(iv) What is the largest possible value of $k$ for this result to hold?

**Exercise 2. Projections**

Let $A^* : H_1 \to H_0$ be the adjoint of a linear operator $A : H_0 \to H_1$, where $H_0$ and $H_1$ are finite-dimensional Hilbert spaces.

(i) Show that $\mathcal{N}(A^*) = \mathcal{R}(A)^\perp$. 

(ii) Let $H_2$ be a finite-dimensional Hilbert space and $B, C : H_2 \to H_0$ two linear operators. Show that $A^* AB = A^* AC$ if and only if $AB = AC$.

(iii) Show that $\mathcal{N}(A^* A) = \mathcal{N}(A)$.

(iv) Show that $\mathcal{R}(A^* A) = \mathcal{R}(A^*)$.

(v) Let $A : H \to H$ be a linear operator on a finite-dimensional Hilbert space $H$. If $A$ is self-adjoint, show that $A$ is an orthogonal projection if and only if $A = A^2$.

Note: for proving any of the given propositions, you are allowed to use the results from previous questions, even if you are not able to prove them.
Exercise 3. Chebyshev Polynomials
We consider the set \( P_2([-1, 1]) \) of all polynomials of degree at most 2 in the vector space \( L^2([-1, 1]) \).

1. Show that \( P_2([-1, 1]) \) is a subspace.

2. The Chebyshev polynomials \( \{T_k\}_{k=0}^2 \) are defined by the following recurrence relation:
   \[
   T_k(t) = 2tT_{k-1}(t) - T_{k-2}(t)
   \]
   \[
   T_0(t) = 1
   \]
   \[
   T_1(t) = t
   \]
   Compute \( T_2(t) \) and show that \( \{T_k\}_{k=0}^2 \) is a basis for \( P_2([-1, 1]) \).

3. We endow the space \( P_2([-1, 1]) \) with the inner product defined as
   \[
   \langle x, y \rangle = \int_{-1}^{1} x(t)y^*(t)dt.
   \]
   Find the dual basis \( \{\tilde{T}_k\}_{k=0}^2 \) for \( \{T_k\}_{k=0}^2 \).
   \[
   \text{Hint: you can use the following:}
   \]
   \[
   \left( \begin{array}{ccc}
   a & 0 & -b \\
   0 & b & 0 \\
   -b & 0 & c \\
   \end{array} \right)^{-1} = \left( \begin{array}{ccc}
   \frac{c}{-b^2 + ac} & 0 & \frac{b}{-b^2 + ac} \\
   0 & \frac{1}{b^2 + ac} & 0 \\
   \frac{b}{-b^2 + ac} & 0 & \frac{-b}{b^2 + ac} \\
   \end{array} \right)
   \]

4. Given \( f(t) = 3t + 4t^2 \), find the value \( \langle f, \tilde{T}_2 \rangle \).

Exercise 4. Inner Product and Construction
In Hilbert space \( (H, \langle ., . \rangle) \), consider the families of \( \{v_n\}_{n \geq 1} \) of norm 1 whose mutual inner products are equal to a constant \( \rho \geq 0 \). Formally, \( \{v_n\}_{n \geq 1} \) is such that for all \( n, m \geq 1 \),
\[
\langle v_n, v_m \rangle = \begin{cases} 1, & \text{if } m = n, \\ \rho, & \text{if } m \neq n. \end{cases}
\]
where \( \rho \) is a real non-negative number. Note that \( \rho = 0 \) is the particular case where \( \{v_n\}_{n \geq 1} \) is orthonormal. Assume that \( \{v_n\}_{n \geq 1} \) satisfies (1).

(i) Why must we have \( \rho \leq 1 \)? When \( \rho = 1 \), show that there is no other possibility than having \( v_n = v_m \) for all \( n, m \geq 1 \).
(ii) Why is (1) equivalent to,
\[
\langle v_n, v_m \rangle = \begin{cases} 1, & \text{if } m = n, \\ \rho, & \text{if } m > n. \end{cases}
\]
(iii) Assuming that \( \rho < 1 \), show for any given \( n \geq 1 \) that the \( n \times n \) matrix of coefficients \( M_{ij} = \langle v_i, v_j \rangle \) where \( 1 \leq i, j \leq n \) is invertible.
(iv) Define
\[ V_n := \text{span}\{v_1, v_2, \cdots, v_n\}. \]

Assuming that \( \rho < 1 \), show that any \( x \in V \) is uniquely characterized by the inner products \( (\langle x, v_i \rangle)_{1 \leq i \leq n} \). Show in the same process that \( (v_1, v_2, \cdots, v_n) \) are independent.

**Exercise 5. Operator Expansion**

Let \( A : H \to H \) be a bounded linear operator with \( \|A\| < 1 \), where \( \|A\| \) denotes the operator norm of \( A \).

(i) Show that \( I - A \) is invertible.

(ii) Show that, for every \( y \) in \( H \),
\[ (I - A)^{-1} y = \sum_{k=0}^{\infty} A^k y. \]

(iii) In practice one can only compute a finite number of terms in the series. For \( \|y\| = 1 \) and \( K \) terms in the expansion, show that the truncation error
\[ \varepsilon_K = \left\| (I - A)^{-1} y - \sum_{k=0}^{K-1} A^k y \right\|. \]

is upper bounded by
\[ \varepsilon_K \leq \frac{\sqrt{\lambda_{\text{max}}}}{1 - \sqrt{\lambda_{\text{max}}}}, \]

where \( \lambda_{\text{max}} \) is the maximum eigenvalue of \( A^* A \).

*Hints: 1) Recast the condition \( \|A\| < 1 \) in terms of eigenvalues and eigenvectors, 2) Operator norm of \( A \) is equal to \( \sqrt{\lambda_{\text{max}}} \).*