Exercise 1. Ideally Matched Sampling and Interpolation

Consider a sampling operator $\tilde{\Phi}^*$ with non-orthonormal rows defined as a linear mapping from $\mathbb{C}^M$ to $\mathbb{C}^N$ with $M > N$. Let $\tilde{S}$ be an orthogonal complement of the nullspace of $\tilde{\Phi}^*$, that is $\tilde{S} = \mathcal{N}(\tilde{\Phi}^*)^\perp$. Choose the interpolating operator to be the pseudoinverse of $\tilde{\Phi}^*$, that is $\Phi = \tilde{\Phi} (\tilde{\Phi}^* \tilde{\Phi})^{-1}$, and denote by $S$ the range space of $\Phi$, $S = \mathcal{R}(\Phi)$. Show that for such a choice, sampling and interpolation are ideally matched, $S = \tilde{S}$, and sampling followed by interpolation is an orthogonal projection.

Exercise 2. Sampling and Interpolation for Bandlimited Vectors

A vector $x \in \mathbb{C}^M$ is called bandlimited when there exists $k_0 \in \{0, 1, \ldots, M-1\}$ such that its DFT coefficient sequence $X$ satisfies

$$X_k = 0 \quad \text{for all } k \text{ with } \frac{k_0+1}{2} \leq k \leq M - \frac{k_0+1}{2}.$$  

(1)

The smallest such $k_0$ is called the bandwidth of $x$. A vector in $\mathbb{C}^M$ that is not bandlimited is called full band. The set of vectors in $\mathbb{C}^M$ with bandwidth at most $k_0$ is a closed subspace. For $x$ in such a bandlimited subspace, find a linear mapping $\Phi : \mathbb{C}^{k_0} \to \mathbb{C}^M$ so that the system described by $\Phi \Phi^*$ achieves perfect recovery $\hat{x} = x$.

Exercise 3. Discrete Sinc Orthonormal Basis for Bandlimited Sequences

Let $N \in \mathbb{Z}_+$, and for each integer $n$ let

$$\varphi_n^k = \frac{1}{\sqrt{N}} \text{sinc}(\frac{\pi}{N}(k - nN)).$$

(i) Prove Parseval’s Equality: If $x$ and $y$ in $\ell^1(\mathbb{Z}) \cap \ell^2(\mathbb{Z})$ have Fourier transforms $X$ and $Y$, then

$$\langle x, y \rangle = \frac{1}{2\pi} \langle X, Y \rangle.$$  

Hint: Express the inner product as a convolution of $x_n$ and $y^*(n)$ evaluated at the origin, and use the convolution theorem.

(ii) Show that the family $\{\varphi_n^k\}_{n \in \mathbb{Z}}$ is orthonormal, that is,

$$\langle \varphi_n^k, \varphi_m^k \rangle = \delta_{n-m}.$$  

(iii) Show that this family forms an orthonormal basis for $BL[-\pi/N, \pi/N]$.

Hint: Let $x \in BL[-\pi/N, \pi/N]$, and $X(e^{j\omega})$ be the Fourier transform of $x$. Observe that $X(e^{j\omega}) = \tilde{X}(\omega)1_{[-\pi/N, \pi/T]}(e^{j\omega})$, where $\tilde{X}(e^{j\omega})$ denotes the periodic extension of $X(e^{j\omega})$. That is,

$$\tilde{X}(e^{j\omega}) = \sum_{l \in \mathbb{Z}} X(e^{j(\omega - l\frac{2\pi}{T})}).$$