Homework 7 (graded)

Thursday, November 24, 2016. Due Thursday, December 8, 2016, 16:15

All exercises are graded and carry 25 points.

Read Sections 5.1 to 5.5 of the book “Foundations of Signal Processing” before starting this homework.

Exercise 1. Fourier series with triangle spectrum

Let \( K \in \mathbb{Z}^+ \), and let \( x \) be the 1-periodic function in \( B_1 \{ -K, ..., K \} \), with Fourier series coefficients:

\[
X_k = \begin{cases} 
1 - |k|/K, & \text{for } |k| \leq K; \\
0, & \text{otherwise.}
\end{cases}
\]

(a) Find a simple expression for \( x \).

(b) What is the largest sampling period \( T_s \) that enables perfect function recovery?

(c) With the sampling period \( T_s \) determined in the previous part, express the condition for perfect recovery in terms of a system of linear equations, by expressing each sample of \( x \) through its Fourier series reconstruction formula.

Exercise 2. Sampling and interpolation

(i) Let \( \tilde{\Phi}^* : \mathbb{R}^4 \to \mathbb{R}^2 \) be a finite-dimensional sampling operator represented in matrix form as

\[
\tilde{\Phi}^* = \begin{bmatrix} 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \end{bmatrix}
\]

(a) What is \( \mathcal{N}(\tilde{\Phi}^*)^\perp \) where \( \mathcal{N} \) stands for the null space?

(b) What is the interpolation operator \( \Phi \) that is ideally matched to \( \tilde{\Phi}^* \)?

(c) Given a vector \( x \in \mathbb{R}^4 \), obtain a formula for \( \hat{x} \in \mathbb{R}^4 \) such that \( \hat{x} \) is of the form

\[
\hat{x} = \begin{bmatrix} a \\
a \\
b \\
\end{bmatrix}
\]

for some \( a, b \in \mathbb{R} \) and \( \| x - \hat{x} \| \) is minimized subject to this constraint. Hint: Use the result from the previous question.

(d) Given a vector \( x \in \mathbb{R}^4 \), obtain a formula for \( \hat{x} \in \mathbb{R}^4 \) such that \( \hat{x} \) is of the form

\[
\hat{x} = \begin{bmatrix} c \\
c + d \\
c + d \\
d \end{bmatrix}
\]

for some \( c, d \in \mathbb{R} \) and \( \| x - \hat{x} \| \) is minimized subject to this constraint. Hint: Follow the same steps as above for an alternative choice of the sampling operator \( \tilde{\Phi}^* \).
Consider the sampling operator $\tilde{\Phi}^* : L^2(\mathbb{R}) \mapsto \ell^2(\mathbb{Z})$ depicted in Figure 1(a) with $T = 1$. Here $x(t)$ is a signal in $L^2(\mathbb{R})$ and $\tilde{g}(t)$ is the box-filter with impulse response given by

$$
\tilde{g}(t) = \begin{cases} 
1, & \text{if } |t| < \frac{1}{2} \\
0, & \text{otherwise}
\end{cases}
$$

(a) What is the interpolation operator $\Phi$ of the form in Figure 1(b) that is ideally matched to $\tilde{\Phi}^*$?

(b) Given a signal $x(t) \in L^2(\mathbb{R})$ identify a piecewise-constant signal $\hat{x}(t)$ such that $\|x - \hat{x}\|$ is minimized subject to the constraint that for every integer $n$ the signal $\hat{x}(t)$ is equal to a constant in the interval $[n, n+1)$. Justify your answer.

Exercise 3. Multichannel sampling

Let $x \in BL[-\pi, \pi]$ and $T = 1$. Consider a generalization of Figure 2 to an $N$-channel system with sampling prefilters $\tilde{g}_i$, $i = 0, 1, ..., N-1$, followed by uniform sampling with period $NT$.

(i) Let $N = 3$, and let the sampling prefilters be derivative filters: $\tilde{G}_i(\omega) = (j\omega)^i$, $i = 0, 1, 2$. Show that the determinant of the matrix $\tilde{G}(\omega)$ is a nonzero constant.

(ii) Let $N \in \mathbb{Z}^+$ and let the sampling prefilters be derivative filters: $\tilde{G}_i(\omega) = (j\omega)^i$, $i = 0, 1, ..., N-1$. Show that the determinant of the matrix $\tilde{G}(\omega)$ is a nonzero constant.

(iii) Let $N = 3$, and let the sampling prefilters be delay filters: $\tilde{G}_0(\omega) = 1$, $\tilde{G}_1(\omega) = e^{-j\omega(1+\alpha)}$, and $\tilde{G}_2(\omega) = e^{-j\omega(1+\beta)}$, with $\alpha, \beta \in [-1, 1]$. For which values of $\alpha$ and $\beta$ is the matrix $\tilde{G}(\omega)$ singular? Numerically compute the condition number $\kappa(\tilde{G}(\omega))$; you should obtain that it does not depend on $\omega$. Plot $\tilde{G}(\omega)$ for $\alpha = 1$ and $\beta \in [-\frac{1}{2}, \frac{1}{2}]$. Comment on the results.

Figure 2: Two-channel sampling and interpolation system. The sampling operator consist of two branches with filtering and sampling in parallel, producing two sampled outputs.
Exercise 4. A Sampling Problem

Assume a sampling setup as follows

where $S$ is a continuous-time system (operator) defined as

$$(Sx)(t) = \int_{t-T/2}^{t+T/2} x(s) \, ds.$$ 

Note that the above system describe a practical sampling and hold system commonly implemented in ADCs.

(i) Give a complete analytic expression for $y_n$

$$y_n = \int \ldots (x \ast h)(t) \, dt.$$ 

(ii) Let $h(t)$ be such that $y(t) = (x \ast h)(t)$. Find and sketch $H(\omega)$.

(iii) Give the necessary condition on $B_0$ and $T$ such that for every $x(t) \in \{\text{functions band-limited to } [-B_0/2, B_0/2]\}$ we have $\hat{x}(t) = x(t)$ (this is independent of the choice of $\Phi$).

(iv) Assuming that the necessary condition from 3) is met, what should $\Phi$ be so that $\hat{x}(t) = x(t)$ for every $x(t) \in \{\text{functions band-limited to } [-B_0/2, B_0/2]\}$?

(that is what is the sufficient condition).

(v) If the conditions in 3) and 4) hold, show that the operator

$$P : L^2(\mathbb{R}) \to \{\text{functions band-limited to } [-B_0/2, B_0/2]\}, \, x \mapsto Px = \hat{x},$$

is an orthogonal projection.

(vi) What should $\Phi$ be so that $\tilde{Q}$ is the adjoint of $Q$? With such a choice of $\Phi$, is $P$ an orthogonal projection?