Homework 8 (graded)

Thursday, December 8, 2016. Due Thursday, December 22, 2016, 16:15

Exercise 1. Computing inner products with splines

Consider a function \( x \in L^2(\mathbb{R}) \) that is zero outside of \([0, \infty)\). We want to compute

\[
\langle x(t), \beta_+(K)(t-n) \rangle_t, \quad n \in \mathbb{Z},
\]

where \( \beta_+(K) \) is the causal elementary B-spline of degree \( K \), using operations on a sequence of samples of the primitive of order \( K + 1 \) of \( x \).

(i) Let \( K = 0 \). Using the definitions of the inner product and of \( \beta_+(0) \) directly, show that

\[
\langle x(t), \beta_+(0)(t-n) \rangle_t = \begin{cases} X_{n+1} - X_n, & \text{for } n \in \mathbb{N}; \\ 0, & \text{otherwise,} \end{cases}
\]

(1a)

where

\[
X_n = \int_0^n x(t) \, dt
\]

is the primitive of \( x \) evaluated at integers.

(ii) Show that (1) holds by using the derivative of \( \beta_+(0) \).

(iii) Let \( \eta(K) \) denote the derivative of order \( K + 1 \) of \( \beta_+(K) \), \( K \in \mathbb{N} \). Show that

\[
\eta(K)(t) = (\delta(t) - \delta(t-1)) * (\delta(t) - \delta(t-1)) * \cdots * (\delta(t) - \delta(t-1)),
\]

(2)

where there are \( K \) convolutions.

(iv) For any \( K \in \mathbb{N} \), let the sequence \( X^{(K)} = x^{(K)}(n), n \in \mathbb{Z} \), be the primitive of order \( K \) of \( x \) evaluated at integers. For any \( K \in \mathbb{N} \), find the filter \( h(K) \) such that

\[
\langle x(t), \beta_+(K)(t-n) \rangle_t = \left( h(K) * X^{(K+1)} \right)_n, \quad n \in \mathbb{Z}.
\]

(3)

Exercise 2. Legendre Polynomials

In this exercise we want to prove the following statement: The Legendre polynomial of degree \( n \) has \( n \) real distinct zeros in the interior of the interval \([-1, 1]\). Note that this is not at all obvious from the definition since it might have multiple or complex zeros, or zeros outside the interval \([-1,1]\).

(i) Show that \( L_n \) has at least one zero in \((-1,1)\) for \( n > 0 \). Hint: it is orthogonal to \( L_0 \).

(ii) Assume that \( t = t_1 \) is a multiple zero of \( L_n \). Argue that \( L_n/(t-t_1)^2 \) is also a polynomial. Obtain a contradiction using again the orthogonality properties of Legendre polynomials. Thus the zeros are distinct.

(iii) Next we want to show that there are indeed \( n \) real zeros. Suppose that \( L_n \) has exactly \( k \geq 1 \) zeros \( t_1, \ldots, t_k \) in \((-1,1)\), and that \( k < n \). We can write

\[
L_n(t) = (t-t_1) \cdots (t-t_k)R(t) = \Pi(t)R(t).
\]

Use the orthogonality condition with the polynomial \( \Pi(t) \) to get to a contradiction. Hint: \( R(t) \) has no zeros in \((-1,1)\).
Exercise 3. Approximating with splines
Recall that \( S_{K,\tau} \) denotes the spline space of degree \( K \) with knots at points in \( \tau \).

(i) Let \( \tau \) denote a subsequence of a strictly-increasing sequence \( \eta \). Show that \( S_{K,\tau} \subset S_{K,\eta} \) for any \( K \in \mathbb{N} \).

(ii) Let \( x \in S_{K_x,\tau_x} \) and \( y \in S_{K_y,\tau_y} \), where \( K_x, K_y \in \mathbb{N} \), and \( \tau_x \) and \( \tau_y \) are strictly-increasing sequences. Does \( x + y \) belong to a spline space in general? Either demonstrate that it may not or describe the spline space that contains \( x + y \).

Hint: Consider the cases \( K_x = K_y \) and \( K_x \neq K_y \) separately, and use the definition of splines.

Exercise 4. Fourier Series, Approximation
Some of the devices used in practice for signal manipulation are far from ideal. For instance, consider the amplification of a signal. Generally, power amplifiers can be well approximated as ideal devices for a certain range of input signal amplitudes. Beyond that range, the amplifier saturates and introduces distortion in the signal. Consider a signal amplifier module as in Figure 1(a) and whose input-output voltage relationship is given in Figure 1(b). For simplicity, let us assume that the gain of the amplifier is \( G = 1 \). Our purpose is to study the effects of such non-linearity in the output signal.

Figure 1: Block diagram of the amplifier (a), and input-output voltage relationship (b). Periodic square waveform with duty cycle \( \tau \) (c).

(i) Compute the Fourier Series coefficients of the 2-periodic square wave \( x_\tau(t) \) with duty cycle \( \tau \), \( 0 < \tau \leq 1 \). The fundamental period of \( x_\tau(t) \) is shown in Figure 1(c).

Suppose that the input signal to the amplifier is a periodized linear B-spline as
\[
x(t) = \beta^{(1)}(t) \star \sum_{k \in \mathbb{Z}} \delta(t - 2k).
\]

(ii) Sketch the output waveform \( y(t) \) (one period) and express it as the convolution of signals of the form of \( x_\tau(t) \). Compute the Fourier series coefficients of the output waveform \( y(t) \).

(iii) Approximate the signal \( y(t) \) using the first \( 2M + 1 \) components of its Fourier series expansion. Provide a bound on the approximation error over one period in the form \( \frac{\gamma}{p(M)} \), where \( \gamma \) is some positive constant and \( p(M) \) is some polynomial of \( M \).